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199 Causal Classes of Space-Time Frames

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It is shown that from the causal point of view Minkowskian space-time admits 199, and only 199, different classes of frames.

1. INTRODUCTION

Space-time is usually described as a four-dimensional Lorentzian manifold. Its topology, its differentiable and metric structures, and its asymptotic properties have been the object of many studies; from the formal point of view there is no doubt that the notion of space-time is at present well defined.

In spite of this fact, a good physical comprehension of this notion has not yet been attained. A point that contributes to this situation is our inability to conceive *directly*[3] local domains of space-time. The importance of this deficiency may be clarified by comparing the evolution of the notion of space-time up to now to the ancient elaboration of the notion of *space*.

For our purposes, this elaboration may be considered as having been achieved after the work of Aristarchus of Samos. The earlier history of the notion of space can be sketched successively as follows[4]: For the Egyptians[5]

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[3]"Directly" means here "without decomposing the local domain into its classical constituents *space and time*."
[4]The variety of mythologies, theologies, and theories among Egyptians and Greeks concerning space is so immense that, obviously, we do not pretend to summarize them in a few lines. We have stripped these sketches from the abundant cosmogonic content in which the corresponding descriptions appear naturally immersed, and we present only those elements attached to the notion of space which have manifestly changed during the period concerned.
[5]We start from Egyptian rather than from Mesopotamian cosmologies because, for our purposes, they do not differ essentially (Eliade, 1976; Roux, 1985; Kramer, 1975).

at about 2500–2300 B.C., the earth is a flat disk crossed by the Nile and surrounded by the sea, and the heaven, separated from the earth by the atmosphere, is supported by eight columns[6]; at 2300–1500 B.C., the scenario is quite similar, but the heaven is supported by four columns.[7] For the Greeks, long before 600 B.C., the earth is a flat disk surrounded by the river Ocean, and the heaven is supported, in the west, by two columns.[8] By about 600 B.C., there are no columns at all, and heaven leans upon the river Ocean, forming the vault of heaven[9]; Anaximander, about 570 B.C., transforms the vault to a celestial sphere and reduces the earth to a relatively small cylinder[10]; Aristarchus of Samos, about 270 B.C., affirms that the earth is spherical and turns around the sun.[11]

The important heuristic point here is that during the first steps, space is constructed by elevation of the heaven *over* the earth, and so it appears as a *ground* ⊕ *height* composed notion. This notion begins to change *asymptotically* with the Vault of Heaven, loses its *global* character with the cylinder of Anaximander, and transforms its basic *ingredients* (ground and height) into simple local frame *parameters* with Aristarchus of Samos.[12] It is only after these changes that the comprehension of the notion of space is attained. Spatial objects are then thought *per se*, without reference to any

[6]These columns, although they frequently appear as divinities (for example, Saumeron *et al.*, 1959), suggest strongly that the heaven is, like the earth from which it has been separated, a flat disc.

[7]The collapse from eight to four columns seems related theologically to the association into pairs of the eight divinities (Grimal, 1988; Grigorieff, 1987) and related visually to the representations of the heaven by the trunk of the goddess Nut (Greenfield Papyrus, British Museum, London; Sarcophagus of Butehamon, Egyptian Museum, Turin) or of the Heavenly Cow (Anthes, 1961); in these last cases, there is no doubt about the identity of the four columns.

[8]Of course, the central role played by the Nile is, for the Greeks, implicitly assumed by the Mediterranean Sea. The two columns indicated in this sketch condense a variety of descriptions involving Atlas and Heracles. Atlas appears bearing the heaven with the aid of columns or directly on his shoulders or on his neck or playing the role of a mountain or even of the Titan who separated the waters of the heaven from those of the earth. Heracles temporarily replaces Atlas in this task during Heracles' labor in the garden of Hesperides, or himself constructs two columns in commemoration of his capture of the oxen of Geryon (Graves, 1967; Desautels, 1988; Ramin, 1979). Curiously, all these "devices" to maintain the heaven seem to be located in the west: Atlas' mountains, Hesperides' garden, Heracles' columns. The mythic and symbolic character of these pictures is magnifically analyzed in Ballabriga (1986). Their scientific absurdity and their lack of balance are, perhaps, at the basis of the conception of the vault of heaven.

[9]This is the conception of the world known by Thales, philosophically modified by him, and transmitted to his pupil Anaximander.

[10]He is the first to speak about antipodes.

[11]In *On the sizes and distances of the sun and moon*; see Heath (1981).

[12]The vanishing of the central role played by the earth in the construction of space is a progressive development; thus, the earth turns around a central fire for Philolaus (about 420 B.C.), and around its axis for Heraclides (about 320 B.C.).

support; this is the case, in particular, for the sphere, which is seen as a realization of total symmetry.[13]

Let us now come back to the notion of space-time. Once Copernicus vindicates the forgotten theory of Aristarchus,[14] the space-time of Galileo and Newton[15] is constructed *over* space and so it appears as space$\oplus$time, a composed notion. The absolute character of its ingredients is lost in Einstein's special relativity, and their local character appears in the general theory. The Penrose conformal-infinity techniques allow us to construct asymptotically some intrinsic concepts not related directly to them, but any time we need a precise physical interpretation, we are *still* constrained to locally decompose space-time into its space$\oplus$time form even though we do not conceive them as "ingredients" but rather as a sort of "comfortable" parametrization.

If we try now to establish a parallel between the evolution of the concepts of space and space-time, it appears that we are at present at a moment analogous to someone situated *after* Anaximander, *but before* Aristarchus: we have not *yet* attained the analog of Aristarchus' development. This last moment would correspond to a *direct* comprehension of space-time, no matter what its decomposition might be.

Is this analogy correct? In other words, is it true that we have not sufficiently integrated the spatial and temporal parts of space-time?[16]

The decomposition of the space-time into space$\oplus$time is intimately related to the use of three rods (space) and one clock (time) to locate space-time events. On the other hand, from a conceptual point of view, clocks and rods are nothing but timelike and spacelike projections of light beams,[17] which can be locally represented respectively by timelike, spacelike, and null directions; thus, the frames associated to the above decomposition of space-time are constituted by one timelike and three spacelike directions. Clearly, they form a *proper* subset of the set of all space-time frames, so that the fact that they are precisely the only frames usually called *physically admissible* is already a sign of the correctness of the above analogy.

[13]It is interesting to note that, after Anaximander, the earth is thought of as a sphere because of a philosophical need for symmetry, not because of observational evidence. The role played by the concept of the sphere in the social structure is considered in Vernant (1988).

[14]The modesty of Copernicus is to his honor; in his *De revolutionibus Caelestibus*, he explicitly refers to Greek astronomers, and especially to Aristarchus (Bonnard, 1959).

[15]We consider here only their kinematic aspects, so that they are identical.

[16]The "integration" we are speaking of here concerns the direct feeling of *hyperbolic* space-time. It has nothing to do with the ability to *use* covariant, intrinsic, or four-dimensional formalisms. These formalisms originate by a more or less direct transcription of the *elliptic* formalism of Riemannian geometry and, in spite of their unquestionable interest, mask (almost) completely the specific features of hyperbolicity.

[17]Think of the present definitions of the units of time and length.

Apart from the physically admissible frames, *how many classes of frames, causally different, does a space-time admit*? The fact that such a natural question has not been asked up to now seems to reinforce our idea that we are still subject to the prejudice of the classical conception of space-time. But it is the absolute lack of intuition about the answer that shows, we believe, the correctness of our analogy: *in 4-dimensional space-time there exist 199 causally different classes of frames.*

What is the interest of such a result?

We have chosen the above historical analogy because, we believe, at the same time it delimits the unripe aspect of the present notion of space-time and it shows intuitively the direction in which it would be developed, which points to the comprehension of every space-time object *per se*, without reference to any spatial support. In order to acquire such an ability, it seems worthwhile to try to develop the habit of regarding space-time objects from as many different viewpoints as possible. The table given below of the 199 different classes of frames thus appears as a basic device for this training.

The possibility seems not too distant of using signals from satellites and planets to perform *solar frames*; this will constrain physicists to study in detail some frames not so "physically admissible" as they have done until now. Our causal classification of them will help this study.

The analysis of the causal classes of frames may suggest new ways to measure the gravitational field. In this direction, among the unusual frames we have already considered, perhaps the more interesting ones are the natural frames attached to what we called *light-coordinates* (Coll, 1985). Roughly speaking, they are local charts such that their four coordinate lines are lightlike geodesics. In principle, they may be constructed in the domain of intersection of four beams of laser light; the four frequencies and the six relative angles between the beams constitute a set of ten quantities which may be related to the ten components of the metric tensor, allowing one to measure it.

Another domain in which the present causal analysis of frames is of interest is the classification of *symmetric frames*. The frames usually employed privilege some space-time directions (the timelike direction from the three spacelike ones in physically admissible frames, the two lightlike directions from the two spacelike ones in null frames). Nevertheless, the cosmological principle suggests in part[18] that some properties of space-time would be best described in such frames that *no direction* be privileged. Such

[18]This is to approach to the Greek adoption of sphericity from symmetry considerations (Vernant, 1988).

frames, constituted by metrically indistinguishable vectors, are called symmetric frames. They have been studied elsewhere from the points of view both of *natural frames* and of *metric-concomitant frames* (Coll and Morales, 1991).

Also, a direct, practical application of the present work is the *taxonomy of local charts.* It allows us to label every local chart with a set of three numbers characterizing the causal class of its associated natural frame (we give examples in Section 4).

Perhaps the more important utility of the causal classification of frames will be found in the study of the deformation of Lorentzian metrics. Indeed, when one performs an arbitrary metric deformation, one obtains a mixed result: a wanted variation of the metric itself and a superflous variation of the field of frames (gauge) with respect to which the metric is expressed. Our results allow us to reduce the group of deformations by considering its "quotient" by the causal classes, that is to say, roughly speaking, by considering nothing but the "199th part of the group" which transforms metrics *but* respects the causal class of the field of frames in which they are expressed.

Anyway, the surprise that the richness of the causal classes of frames has produced in all of us shows certainly that we have not yet attained the intellectual right to write the word *space-time* without its hyphen.

The paper is organized as follows: In Section 2 we consider, for the sake of simplicity, some general notions in arbitrary dimensions. In Section 3 we expose a set of arguments allowing us to deduce the existence of the 199 classes announced. Finally, in Section 4 we present the corresponding table of causal classes and comment on some applications.

The table of the causal classes of frames was presented previously, without proof (Coll and Morales, 1988, 1989).

2. *n*-DIMENSIONAL ASPECTS

(a) Let $\mathbf{r}$ denote a *frame* of a linear space E_n, that is, an *ordered* basis of vectors, $\mathbf{r}\equiv\{e_\alpha\}$, $\alpha\in I_n\equiv\{1,\ldots,n\}$, and let J_p be any of the $\binom{n}{p}$ combinations of p elements of I_n, $1\leq p<n$. The p-planes Π_p of E_n generated by p elements of $\mathbf{r}$, $\Pi_p\equiv\{\lambda^h e_h \mid h\in J_p\}$, $\lambda^h\in\mathbb{R}$, are the *adjoint p-planes* of $\mathbf{r}$. Let $\Lambda\mathbf{r}$ be a *homothetic deformation* of $\mathbf{r}$, $\Lambda\mathbf{r}\equiv\{e'_\alpha \mid e'_\alpha\equiv\lambda_\alpha e_\alpha\}$, $\lambda_\alpha\in\mathbb{R}-\{0\}$, and $\Theta\mathbf{r}$ a permutation of $\mathbf{r}$, $\Theta\mathbf{r}\equiv\{e_{\alpha'} \mid \alpha'=\theta(\alpha)\}$, $\theta(\alpha)$ being a permutation of I_n. Two frames $\mathbf{r}$ and $\mathbf{r}'$ have the same adjoint p-planes, for any p, if and only if $\mathbf{r}'=\Theta\Lambda\mathbf{r}$.

Let Π_s and Π'_s be two s-planes corresponding, respectively, to the combinations $J_s=\{\sigma_1,\ldots,\sigma_s\}$ and $J'_s=\{\sigma'_1,\ldots,\sigma'_s\}$, where $\sigma_1<\cdots<\sigma_s$

and $\sigma'_1 < \cdots < \sigma'_s$. We shall say that Π_s *precedes* Π'_s if there exists t such that $\sigma_1 = \sigma'_1, \ldots, \sigma_{t-1} = \sigma'_{t-1}, \sigma_t < \sigma'_t$. Thus, the *adjoint set* of all the s-planes of $\mathbf{r}$,

$$\Pi(\mathbf{r}) \equiv \bigcup_{s=1}^{n-1} \Pi_s$$

is an ordered set of $2^n - 2$ elements.

(b) Suppose now E_n is endowed with a hyperbolic metric g of arbitrary signature. The *causal type* of an s-plane is timelike, null, or spacelike if the restriction of g to it is respectively hyperbolic, degenerate, or elliptic. The *causal character* of the adjoint set $\Pi(\mathbf{r})$ is the ordered sequence of the causal types of the adjoint s-planes of $\mathbf{r}$. Let $\mathbf{r}$ and $\mathbf{r}'$ be two frames with adjoint sets $\Pi(\mathbf{r})$ and $\Pi(\mathbf{r}')$, respectively.

Definition. The frames $\mathbf{r}$ and $\mathbf{r}'$ belong to the same *causal class* if there exists a permutation Θ such that $\Pi(\mathbf{r})$ and $\Pi(\Theta\mathbf{r}')$ have the same causal character.

Denote by $[\mathbf{r}]$ the causal class of $\mathbf{r}$, let $\{\theta^\alpha\}$ be the algebraic dual coframe of $\mathbf{r} = \{e_\alpha\}$, $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$, and let $\theta_\alpha \equiv g(\theta^\alpha)$ be the vectors associated to θ^α by g. The causal class $[\mathbf{r}^*]$ of the frame $\mathbf{r}^* \equiv \{\theta_\alpha\}$ is called the *dual causal class* of $[\mathbf{r}]$. If $[\mathbf{r}^*] = [\mathbf{r}]$, $[\mathbf{r}]$ is said to be *self-dual.*

The adjoint s-plane of $\mathbf{r}^*$ associated to the combination $J_s = \{\sigma_1, \ldots, \sigma_s\}$ is orthogonal to the $(n-s)$-plane associated to the combination $J_{n-s} \equiv I_n - J_s$ and we have:

Proposition 1. The causal class $[\mathbf{r}]$ of a frame $\mathbf{r}$ is determined by the sequence of the causal types of the adjoint s-planes of $\mathbf{r}$ and of the causal types of the adjoint s'-planes of $\mathbf{r}^*$ where $1 \le s \le n-k$ and $s' \le k-1$ for any integer $k \le n-1$.

(c) It is known that the hyperbolic type (p, q) of a metric g, $p+q=n$, is determined by its signature, $\sigma(g) \equiv p-q$. In a similar way, we can associate to every frame $\mathbf{r}$ a *causal signature* $\sigma(\mathbf{r})$ which determines the number of vectors of the frame which are timelike, null, or spacelike.

Let $\mathbf{r}$ be a frame constituted by p timelike, q null, and r spacelike vectors, $p+q+r=n$; the triplet (p, q, r) is called the *causal type* of $\mathbf{r}$. On the set of causal types, we define the following order: (p, q, r) *precedes* (p', q', r') if $r > r'$ or $r = r'$ and $q > q'$. The ordinal of the causal type will be called the *causal signature* σ of $\mathbf{r}$. We have:

Proposition 2. The causal signature σ of a frame $\mathbf{r}$ of causal type (p, q, r) is given by

$$\sigma = \tfrac{1}{2}(p+q)(p+q+1) + p + 1$$

Conversely, the causal type can be obtained from the causal signature. Taking into account that, for a given σ, $s \equiv p+q$ is the highest integer satisfying $s^2+s+2(1-\sigma)\leq 0$, we obtain the following:

Proposition 3. Let σ be the causal signature of a frame; its causal type (p, q, r) is given by $p=\sigma-1-s(s+1)/2$, $q=s-p$, $r=n-s$, where

$$s=\mathbb{E}(\tfrac{1}{2}[(8\sigma-7)^{1/2}-1])$$

$\mathbb{E}$ is the integral part function.

It is clear that σ is an integer which satisfies $1\leq\sigma\leq\binom{n+2}{2}$. In particular, the causal signatures $\sigma=1$, $\sigma=1+n(n+1)/2$, and $\sigma=(n+1)(n+2)/2$ correspond to frames whose vectors are, respectively, spacelike, null, and timelike, that is, of causal types $(0, 0, n)$, $(0, n, 0)$, and $(n, 0, 0)$, respectively. The *normal* frames of causal type $(1, 0, n-1)$, which are the generalization to n dimensions of the physically admissible frames of the space-time, have causal signature $\sigma=3$, and the *null* frames of causal type $(0, 2, n-2)$ have $\sigma=4$.

Let us note that the order we have assigned to the causal types induces an interesting property of *dimensional invariance*: as shown by Proposition 2, the causal signature of a frame of causal type $(p, q, \cdot)$ is independent of the dimension n of the space.

(d) From now on, we consider E_n endowed with a *Lorentzian* metric g; we have the following simple lemmas:

Lemma 1. Let us consider s linearly independent directions. (i) If they are spacelike, they generate an s-plane that can be spacelike, null, or timelike. (ii) If one of them is null and the others are spacelike, they generate an s-plane that can be null or timelike. (iii) If one direction is timelike or two of them are null, they generate a timelike s-plane.

Lemma 2. The null direction of a Lorentzian frame of causal signature $\sigma=2$ cannot be orthogonal to the other $n-1$ spacelike directions.

Consider now the dual frame $r^*=\{\theta_\alpha\}$ of $\mathbf{r}$; for every α, θ_α is orthogonal to the adjoint $(n-1)$-plane Π_{n-1} of $\mathbf{r}$ corresponding to the combination $I_n-\{\alpha\}$. Because the preceding lemmas and the fact that θ_α is respectively spacelike, null, or timelike according to the timelike, null, or spacelike character of Π_{n-1}, the causal type of $\mathbf{r}^*$ is partially related to the causal type of $\mathbf{r}$. Thus, if $\sigma(\mathbf{r})=1$, all the adjoint hyperplanes of $\mathbf{r}$ are generated by spacelike vectors and $\mathbf{r}^*$ may have any causal character. If $\sigma(\mathbf{r})=2$, only one adjoint hyperplane is generated by spacelike vectors; the others are generated by $n-2$ spacelike vectors and one null vector and, from Lemma 2, these $n-1$ hyperplanes are either one null and the others timelike, or they are all timelike. If $\sigma(\mathbf{r})=3$, there are $n-1$ timelike adjoint hyperplanes; the other one is generated by spacelike vectors. If $\sigma(\mathbf{r})=4$, all the adjoint hyperplanes are nonspacelike and at most two of them are null. If $\sigma(\mathbf{r})=5$, the causal type of $\mathbf{r}$ is $(1, 1, n-2)$ and, consequently, there are $n-1$ timelike adjoint hyperplanes; the other one is either timelike or null. If $\sigma(\mathbf{r})=6$, there are two timelike vectors; and if $\sigma(r)>6$, at least three vectors are nonspacelike. Consequently, for $\sigma(\mathbf{r})>5$ all the adjoint hyperplanes are timelike and hence $\sigma(\mathbf{r}^*)=1$. We have thus shown:

Proposition 4. Let $\mathbf{r}$ be a Lorentzian frame in dimension n.
If $\sigma(\mathbf{r})=1$, then $\sigma(\mathbf{r}^*)=1, 2, \ldots, (n+1)(n+2)/2$.
If $\sigma(\mathbf{r})=2$, then $\sigma(\mathbf{r}^*)=1, 2, 3, 4, 5$.
If $\sigma(\mathbf{r})=3$, then $\sigma(\mathbf{r}^*)=1, 2, 3$.
If $\sigma(\mathbf{r})=4$, then $\sigma(\mathbf{r}^*)=1, 2, 4$.
If $\sigma(\mathbf{r})=5$, then $\sigma(\mathbf{r}^*)=1, 2$.
If $\sigma(\mathbf{r})>5$, then $\sigma(\mathbf{r}^*)=1$.

(e) When $\sigma(\mathbf{r})=\sigma(\mathbf{r}^*)=2$ the frames are of causal type $(0, 1, n-1)$, but they may present different properties:

(α) Their spacelike vectors generate a null hyperplane, all the others being timelike; in this case the null vectors of $\mathbf{r}$ and $\mathbf{r}^*$ cannot be collinear.

(β) Their spacelike vectors generate a timelike hyperplane in such a way that $n-2$ of these vectors are orthogonal to the null vector of the frame; in this case the null vectors of $\mathbf{r}$ and $\mathbf{r}^*$ are necessarily collinear.

Except for these two cases, all causal properties of the vectors and adjoint hyperplanes of $\mathbf{r}$ are completely determined by $\sigma(\mathbf{r})$ and $\sigma(\mathbf{r}^*)$.

It is clear that two frames $\mathbf{r}$ and $\mathbf{r}'$, which differ by a permutation Θ, $\mathbf{r}'=\Theta\mathbf{r}$, belong to the same causal class. Let us denote by $\mathbf{t}$, $\mathbf{i}$, and $\mathbf{e}$, respectively, timelike, null, and spacelike vectors; by a permutation, we may associate, to every frame $\mathbf{r}$ of causal character (p, q, r), an *ordered frame* $o(\mathbf{r})\equiv\{\mathbf{t}_1, \ldots, \mathbf{t}_p, \mathbf{i}_1, \ldots, \mathbf{i}_q, \mathbf{e}_1, \ldots, \mathbf{e}_r\}$, $\mathbf{t}_i$, $\mathbf{i}_j$, $\mathbf{e}_k$ belonging to $\mathbf{r}$. Obviously, the ordered frames associated to the frame $\mathbf{r}$ are determinated up to a permutation Θ of the form $\Theta=\theta_p\times\theta_q\times\theta_r$, where θ_i denotes a permutation of i elements. In general, the dual $o^*(\mathbf{r})$ of an ordered frame $o(\mathbf{r})$ is not an ordered frame: $o^*(\mathbf{r})\neq o(\mathbf{r}^*)$. But one can show that, apart from the frames belonging to the above case β, there exist ordered frames, which will be noted by $c(\mathbf{r})$, such that $c^*(\mathbf{r})=c(\mathbf{r}^*)$. In the exceptional case β, one can always find ordered frames such that their duals are of the form $\{\mathbf{e}_1, \mathbf{i}, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}\}$: they will also be denoted by $c(\mathbf{r})$ and, like the ones satisfying $c^*(\mathbf{r})=c(\mathbf{r}^*)$, will be called *canonically ordered frames.*

From these considerations and Proposition 4, we can show the following:

Proposition 5. For an n-dimensional Lorentzian metric, the number N of pairs of canonically ordered frames $\{c(\mathbf{r}), c(\mathbf{r}^*)\}$ having different causal characters is given by $N=(n+1)(n+2)+9$.

Table I. Correspondence Between the Causal Character of n-Dimensional Lorentzian Frames and Their Duals.[a]

σ \ σ^*	1^*	2^*	2^*_β	3^*	4^*	5^*	6^*	...	l^*	...
1	π_1^1	π_2^1		π_3^1	π_4^1	π_5^1	π_6^1	...	π_l^1	...
2	π_1^2	π_2^2	$\pi_{2_\beta}^2$	π_3^2	π_4^2	π_5^2				
3	π_1^3	π_2^3		π_3^3						
4	π_1^4	π_2^4			π_4^4					
5	π_1^5	π_2^5								
6	π_1^6									
...	...									
m	π_1^m									
...	...									

[a]The frames are ordered according to their causal signature σ. The π_l^m stand for the sets of adjoint s-planes of the frame, $1 < s < n-1$, and the column 2^*_β corresponds to the exceptional β case mentioned in the text.

(f) Denoting, for brevity by σ^* the causal signature of $\mathbf{r}^*$, $\sigma^* \equiv \sigma(\mathbf{r}^*)$, we have that the table of the N different pairs $\{c(\mathbf{r}), c(\mathbf{r}^*)\}$ adopts the aspect indicated in Table I. In it, one has $l, m \leq \binom{n+2}{2}$, and π_l^m denotes the set of causal types of s-planes, $1 < s < n-1$. The table follows from Propositions 1, 4, and 5. The above-mentioned property of dimensional invariance induced by the chosen order is here clearly apparent: for increasing n, the occupied cases remain occupied, and the only occupied cases to be added are of the form π_l^1 and π_1^m. Of course, the *number* of s-planes contained in every case depends, in general, on the dimension n. Its evaluation for $n = 4$ will be our task in the next section.

Remembering Proposition 1, we have that the next result follows directly from Proposition 5:

Corollary. In dimension 3, there exist 29 causal classes of Lorentzian frames.

3. THE CAUSAL CLASSIFICATION OF SPACE-TIME FRAMES

(a) For $n = 4$, we can distinguish $N = 39$ causally different pairs $\{c(\mathbf{r}), c^*(\mathbf{r})\}$. From Proposition 1, a complete causal study of the space-time frames still requires us to specify the causal types of the adjoint 2-planes corresponding to every one of these 39 pairs.

Let us consider the 2-plane π_{xy} generated by the vectors x and y. The sign ε of the quantity $g(x, x)g(y, y) - [g(x, y)]^2$ depends neither on the choice of the basis on π_{xy} nor on the sign of the signature of the Lorentzian metric g. It will be called the *causal sign* of the 2-plane π_{xy} since we have

$\varepsilon = +$, $\varepsilon = 0$, or $\varepsilon = -$, depending on whether π_{xy} is spacelike, null, or timelike.

Let $\pi_{\alpha\beta}$ be the adjoint 2-plane of the frame $\mathbf{r} = \{e_\alpha\}$ generated by e_α and e_β, and let $\varepsilon_{\alpha\beta}$ be its causal sign. Denoting by $\varepsilon^*_{\alpha\beta}$ the causal sign of the adjoint 2-plane $\pi^*_{\alpha\beta}$ generated by the vectors θ_α and θ_β of r^*, we obtain the following result:

Proposition 6. For any distinct values of the indices α, β, γ, and δ, the causal signs $\varepsilon_{\alpha\beta}$ and $\varepsilon^*_{\gamma\delta}$ are related by $\varepsilon_{\alpha\beta} = -\varepsilon^*_{\gamma\delta}$.

Thus, if $\{\varepsilon_{12}\,\varepsilon_{13}\,\varepsilon_{14}\,\varepsilon_{23}\,\varepsilon_{24}\,\varepsilon_{34}\}$ is the ordered set of causal signs of the set Π of the adjoints 2-planes of $\mathbf{r}$, the corresponding set of $\mathbf{r}^*$ is given by $\{-\varepsilon_{34} - \varepsilon_{24} - \varepsilon_{23} - \varepsilon_{14} - \varepsilon_{13} - \varepsilon_{12}\}$.

The invariance group, say Θ_c, of the pair $\{c(\mathbf{r}), c^*(\mathbf{r})\}$ does not respect, in general, the order of the causal characters of the adjoint 2-planes; that is, the whole class of frames $c(\mathbf{r})$ is too large to be used to distinguish causal classes. From now on, we shall restrict $c(\mathbf{r})$ in such a way that those of the causal signs that are not invariant by the action of Θ_c are ordered nondecreasingly (i.e., $-$, 0, $+$). The set of adjoint 2-planes of $c(\mathbf{r})$ *so ordered* will be denoted by $c(\pi)$.

Now, we are able to study the different causal classes of frames. The method works in two steps: first, one obtains the sets $c(\pi)$ associated to a given $c(\mathbf{r})$, and second, one checks the sets $c^*(\pi)$ corresponding to all the possible $c^*(\mathbf{r})$.

(b) Causal classes with $\sigma(\mathbf{r}) = 1$. Since $c(\mathbf{r}) = \{\mathbf{eeee}\}$, the adjoint 2-planes of $\mathbf{r}$ may have any causal type. If $\sigma(\mathbf{r}^*) = 1$, all the causal characters of $c(\mathbf{r})$ may be permuted: $\Theta_c \approx \theta_4$. Therefore, all the signs of any $c(\pi)$ may be ordered in a non-decreasing way. The $c(\pi)$, considered as frames of the 6-dimensional bivector space, may be ordered by their causal signs; the result is

$$\begin{array}{llll}
\{++++++\}, & \{0+++++\}, & \{-+++++\}, & \{00++++\}, \\
\{-0++++\}, & \{--++++\}, & \{000+++\}, & \{-00+++\}, \\
\{--0+++\}, & \{---+++\}, & \{0000++\}, & \{-000++\}, \\
\{--00++\}, & \{---0++\}, & \{----++\}, & \{00000+\}, \\
\{-0000+\}, & \{--000+\}, & \{---00+\}, & \{----0+\}, \\
\{-----+\}, & \{000000\}, & \{-00000\}, & \{--0000\}, \\
\{---000\}, & \{----00\}, & \{-----0\}, & \{------\}
\end{array}$$

Thus, there are 28 causal classes with $\sigma(\mathbf{r}) = \sigma(\mathbf{r}^*) = 1$. For $\sigma(\mathbf{r}) = 1$ and

$\sigma(\mathbf{r}^*) \geq 2$, the corresponding causal classes are the dual of the causal classes with $\sigma(\mathbf{r}) \geq 2$ and $\sigma(\mathbf{r}^*) = 1$. These will be obtained below.

(c) Causal classes with $\sigma(\mathbf{r}) = 2$. Now $c(\mathbf{r}) = \{\mathbf{ie}_1\mathbf{e}_2\mathbf{e}_3\}$ and, from Lemma 2, the three adjoint 2-planes $(\mathbf{ie}_1)$, $(\mathbf{ie}_2)$, $(\mathbf{ie}_3)$ cannot be null at once. Either (1) they are timelike or (2) two of them are timelike and the other one is null, or (3) only one is timelike and the others are null. Denoting by Δ any causal character (that is, $\Delta = \mathbf{t}, \mathbf{i}, \mathbf{e}$ for vectors, and $\Delta = -, 0, +$ for 2-planes), we have that the cases 1 and 2 correspond to $c^*(\mathbf{r}) = \{\Delta\mathbf{eee}\}$ since the adjoint hyperplanes $\mathcal{H}_1 \equiv (\mathbf{ie}_2\mathbf{e}_3)$, $\mathcal{H}_2 \equiv (\mathbf{ie}_1\mathbf{e}_3)$, and $\mathcal{H}_3 \equiv (\mathbf{ie}_1\mathbf{e}_2)$ are timelike. For them we have $\Theta_c \approx \theta_3$. Let us choose $\mathbf{e}_1$ and $\mathbf{e}_2$ in such a way that the first and the second adjoint 2-planes of $\mathbf{r}$ are timelike. We have then $\{---\Delta\Delta\Delta\}$ for case 1 and $\{--0\Delta\Delta\Delta\}$ for case 2.

For case 1, no spacelike vector of $\mathbf{r}$ has been privileged, so that we can take the 2-planes $[(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3)(\mathbf{e}_2\mathbf{e}_3)]$ of the hyperplane $\mathcal{H}_4 \equiv (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)$ with their causal signs in a nondecreasing order:

$$[---] \quad [--0] \quad [--+] \quad [-00] \quad [-0+]$$

$$[-++] \quad [000] \quad [00+] \quad [0++] \quad [+++]$$

For case 2, the vector $\mathbf{e}_3$ remains privileged with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$, which are still interchangeable. For every causal type of the 2-plane $(\mathbf{e}_1\mathbf{e}_2)$, we can take the two 2-planes $(\mathbf{e}_1\mathbf{e}_3)$ and $(\mathbf{e}_2\mathbf{e}_3)$ with their causal signs in a nondecreasing order. Now, in terms of their signs, the adjoint 2-planes of $\mathcal{H}_4$ are

$$[\Delta--] \quad [\Delta-0] \quad [\Delta-+] \quad [\Delta 00] \quad [\Delta 0+] \quad [\Delta++]$$

For cases 1 and 2, there are 28 different sets $c(\pi)$. The corresponding dual sets $c^*(\pi)$ are obtained from Proposition 6. Now, taking into account Lemma 1, it remains to check the sets $c^*(\pi)$ that are compatible with every one of the three sets $c^*(\mathbf{r}) = \{\Delta\mathbf{eee}\}$. Of course, if $c^*(\mathbf{r}) = \{\mathbf{eeee}\}$ there are no additional restrictions.

If $c^*(\mathbf{r}) = \{\mathbf{teee}\}$, the first three signs in $c^*(\pi)$ are negative. So, the last three signs in $c(\pi)$ are positive. The possible sets $c(\pi)$ are $\{---+++\}$ and $\{--0+++\}$.

If $c^*(\mathbf{r}) = \{\mathbf{ieee}\}$, the first three signs in $c^*(\pi)$ are nonpositive and simultaneously nonzero; this implies the following possibilities for $c^*(\pi)$: $\{-00+++\}$, $\{--0+++\}$, $\{---+++\}$, $\{-000++\}$, $\{--00++\}$, $\{00-0++\}$, $\{-0-0++\}$, and $\{---0++\}$. But $\{-00+++\}$ and $\{-000++\}$ are forbidden by the following simple lemma.

Lemma 3. If $c^*(\mathbf{r}) = \{\mathbf{ieee}\}$ and $c^*(\pi) = \{-00\Delta\Delta+\}$, then $c(\mathbf{r}) = \{\Delta\mathbf{iee}\}$.

Now $c^*(\pi)=\{00-0++\}$ is not compatible with $c^*(\mathbf{r})$: since the first two adjoint 2-planes are null, the fourth must be spacelike. In consequence, if $c(\mathbf{r})=\{\mathbf{ieee}\}$ and $c^*(\mathbf{r})=\{\mathbf{ieee}\}$, then

$$c(\pi):\ \{---0++\}\ \ \{---+++\}\ \ \{--00++\}\ \ \{--0+0+\}\ \ \{--0+++\}$$

Finally, let us consider case 3, that is, the case when one of the adjoint hyperplanes $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$ is null (the others being necessarily timelike). The frame $\mathbf{r}^*$ also contains the null direction of $\mathbf{r}$. Suppose $\mathscr{H}_3=(\mathbf{i}\mathbf{e}_2\mathbf{e}_3)$ is the null hyperplane; this fixes the first spacelike vector of $\mathbf{r}$: the adjoint 2-plane $(\mathbf{i}\mathbf{e}_1)$ is timelike. Since the adjoint 2-plane $(\mathbf{e}_2\mathbf{e}_3)$ is spacelike, we have $c(\pi)=\{-00\Delta\Delta+\}$. The fourth and the fifth signs of $c(\pi)$ are interchangeable, that is, $\Theta_c\approx\theta_2$; setting them in a nondecreasing order, we have

$$c(\pi):\ \{-00--+\}\ \ \{-00-0+\}\ \ \{-00-++\}$$
$$\{-0000+\}\ \ \{-000++\}\ \ \{-00+++\}$$

From Proposition 6 it then follows that

$$c^*(\pi):\ \{-++00+\}\ \ \{-0+00+\}\ \ \{--+00+\}$$
$$\{-0000+\}\ \ \{--000+\}\ \ \{---00+\}$$

which are all compatible with $c^*(\mathbf{r})=\{\mathbf{eiee}\}$. If $c^*(\mathbf{r})=\{\mathbf{tiee}\}$, only $c^*(\pi)=\{---00+\}$ is possible. If $c^*(\mathbf{r})=\{\mathbf{iiee}\}$, the sets $c^*(\pi)$ neither contain the sign plus in the first three places nor are of the form $c(\pi^*)=\{-0000+\}$, because then the second vector of $\mathbf{r}$ would be null, in contradiction with $c(\mathbf{r})=\{\mathbf{ieee}\}$. Consequently, the only possible $c^*(\pi)$ are $\{--000+\}$ and $\{---00+\}$.

(d) Causal classes with $\sigma(\mathbf{r})=3$. We have $c(\mathbf{r})=\{\mathbf{teee}\}$ and $c(\pi)=\{---\Delta\Delta\Delta\}$. From Proposition 4, one has $\sigma(\mathbf{r}^*)=1,2,3$, that is, $c^*(\mathbf{r})=\{\Delta\mathbf{eee}\}$ and consequently $\Theta_c\approx\theta_3$. Thus, the signs $[\Delta\Delta\Delta]$ can always be ordered in a nondecreasing way,

$$c(\pi):\begin{Bmatrix}\{------\} & \{-----0\} & \{-----+\} & \{----00\} & \{----0+\}\\ \{----++\} & \{---000\} & \{---00+\} & \{---0++\} & \{---+++\}\end{Bmatrix}$$

and their respective duals are

$$c^*(\pi):\begin{Bmatrix}\{++++++\} & \{0+++++\} & \{-+++++\} & \{00++++\} & \{-0++++\}\\ \{--++++\} & \{000+++\} & \{-00+++\} & \{--0+++\} & \{---+++\}\end{Bmatrix}$$

If $c^*(\mathbf{r})=\{\mathbf{eeee}\}$, all the above $c^*(\pi)$ are possible by Lemma 1. If $c^*(\mathbf{r})=\{\mathbf{teee}\}$, only $c^*(\pi)=\{---+++\}$ is possible. And, if $c^*(\mathbf{r})=\{\mathbf{ieee}\}$, the possible $c^*(\pi)$ are $\{---+++\}$ and $\{--0+++\}$. In fact, the three first adjoint 2-planes of $c^*(\mathbf{r})$ are simply restricted to be not spacelike, but

$c^*(\pi)=\{000+++\}$ is forbidden by Lemma 2 and $c^*(\pi)=\{-00+++\}$ cannot occur by Lemma 3.

(e) Causal classes with $\sigma(\mathbf{r})=4$. Now, $c(\mathbf{r})=\{\mathbf{i}_1\mathbf{i}_2\mathbf{e}_1\mathbf{e}_2\}$ and the causal types of the adjoint 2-planes are of the form $\{-\circ\circ\circ\circ\Delta\}$, where $\circ$ stands for $-$ or 0. There are the following possibilities:

1. $\{-----\Delta\}$
2. $\{----0\Delta\}$, $\{---0-\Delta\}$ (θ_e), $\{--0--\Delta\}$ (θ_i), $\{-0---\Delta\}$ $(\theta_i\times\theta_e)$
3. $\{--0-0\Delta\}$, $\{-0-0-\Delta\}$ (θ_e)
4. $\{--00-\Delta\}$, $\{-0--0\Delta\}$ (θ_i) or (θ_e)
5. $\{---00+\}$, $\{-00--+\}$ (θ_i)
6. $\{--000+\}$, $\{-0-00+\}$ (θ_e), $\{-00-0+\}$ (θ_i), $\{-000-+\}$ $(\theta_i\times\theta_e)$
7. $\{-0000+\}$

Every one of these rows corresponds, for every value of Δ, to the same causal class. The first term on every row has already the correct order, so that it is the causal sign representation of the corresponding $c(\pi)$. Also, we write the permutation going from every causal configuration of adjoint 2-planes to the ordered set $c(\pi)$; thus, θ_i (resp. θ_e) is the transposition of the null (resp. spacelike) vectors of $\mathbf{r}$. In 5–7 the adjoint 2-plane $(\mathbf{e}_1\mathbf{e}_2)$ is spacelike since both vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal to the same null direction. If $c^*(\mathbf{r})=\{\mathbf{eeee}\}$, the adjoint hyperplanes $\mathscr{H}_1\equiv(\mathbf{i}_2\mathbf{e}_1\mathbf{e}_2)$ and $\mathscr{H}_2\equiv(\mathbf{i}_1\mathbf{e}_1\mathbf{e}_2)$ are both timelike, and the adjoint 2-planes are $\{-----\Delta\}$, $\{----0\Delta\}$, $\{--0-0\Delta\}$, and $\{--00-\Delta\}$. If $c^*(\mathbf{r})=\{\mathbf{i}_2\mathbf{eee}\}$, then $\mathscr{H}_1$ is null and $\mathscr{H}_2$ is timelike. In this case, there are two possibilities for $c(\pi)$: $\{---00+\}$ and $\{--000+\}$. If $c^*(\mathbf{r})=\{\mathbf{i}_2\mathbf{i}_1\mathbf{ee}\}$, then $c(\pi)=\{-0000+\}$ due to the fact that $\mathscr{H}_1$ and $\mathscr{H}_2$ are null. Note that the null directions of $\mathbf{r}$ and $\mathbf{r}^*$ are the same, but their order is interchanged.

(f) Causal classes with $\sigma(\mathbf{r})=5$. Now, $c(\mathbf{r})=\{\mathbf{tie}_1\mathbf{e}_2\}$ and the three first adjoint 2-planes of $\mathbf{r}$ are timelike. The adjoint hyperplane $\mathscr{H}_1\equiv(\mathbf{ie}_1\mathbf{e}_2)$ is null or timelike. If $\mathscr{H}_1$ is null, then $c(\pi)=\{---00+\}$ and $c(\mathbf{r}^*)=\{\mathbf{ieee}\}$ (with the same null direction as $\mathbf{r}$). If $\mathscr{H}_1$ is timelike, that is, $c(\mathbf{r}^*)=\{\mathbf{eeee}\}$, then the causal characters of the adjoint 2-planes $[(\mathbf{ie}_1)(\mathbf{ie}_2)(\mathbf{e}_1\mathbf{e}_2)]$ are $[--\Delta]$ or $[-0\Delta]$. In the latter case we can set the two first signs in increasing order due to the fact that Θ_c is now the transposition of $\mathbf{e}_1$ and $\mathbf{e}_2$.

(g) Causal classes with $\sigma(\mathbf{r})>5$. From Proposition 4, $c(\mathbf{r}^*)=\{\mathbf{eeee}\}$, and these causal classes are obtained directly as follows. If $\sigma(\mathbf{r})=6$, then $c(\mathbf{r})=\{\mathbf{ttee}\}$ and $c(\pi)=\{-----\Delta\}$. If $\sigma(\mathbf{r})=7$, then $c(\mathbf{r})=\{\mathbf{iiie}\}$ and $c(\pi)=\{--\circ-\circ\circ\}$. Because of $\Theta_c\approx\theta_3$ the signs $[\circ\circ\circ]$ can be placed in nondecreasing order. This gives four causal classes. If $\sigma(\mathbf{r})=8$, then $c(\mathbf{r})=\{\mathbf{tiie}\}$, $c(\pi)=\{----\circ\circ\}$, and there are three causal classes. If $\sigma(\mathbf{r})=9$, then $c(\mathbf{r})=\{\mathbf{ttie}\}$

and $c(\pi)=\{-----\circ\}$. If $\sigma(\mathbf{r})=10, 11, 12, 13, 14$, or 15, then all the adjoint 2-planes of $\mathbf{r}$ are timelike, that is, $c(\pi)=\{------\}$.

(h) Using the preceding results and counting the different possibilities, we have the following result.

Theorem. Space-time admits 199, and only 199, causal classes of frames.

4. DISCUSSION AND COMMENTS

(a) The considerations of the preceding section not only lead to the above theorem, but also allow us to construct explicitly the characterization of all the causal classes. This characterization is given in Table II. Table II differs from Table I in that it is a particularization to dimension $n=4$; in that it makes explicit the notation of the causal character of the frames $\mathbf{r}$ and $\mathbf{r}^*$ (remember that, as shown by Proposition 3, there is a *bijection* between the causal character of $\mathbf{r}$ and its causal signature); in that it splits the cases corresponding to the pairs $\{\mathbf{r}, \mathbf{r}^*\}$ [by giving them in convenient order we may identify $\mathbf{r}$ and $c(\mathbf{r})$]; and in that it makes explicit the notation of the causal character of the adjoint 2-planes.

The natural reading of Table II begins from the left. For example, let us be given a frame of causal type {**iiee**}; the corresponding row of the table indicates that it may belong to $3\times 4+2+1=15$ causal classes. If, in addition, we know that its dual is of causal type {**ieee**}, the intersection of the row with the corresponding column of duals restricts the number of classes to two. They correspond to the only possibilities, for the adjoint 2-plane of the first and last vectors of $\mathbf{r}$, of being timelike or null.

(b) Thus, we can see some simple properties: (i) There are *impossible Lorentzian frames* (blank cases of the table); for example, there is no frame {**teee**} having as dual a coframe {**eiee**}. (ii) There exist only six causal classes which may be *univocally* determined by the causal character of the frame (the six last classes of the first column). (iii) Only the frames of causal character {**eeee**} with dual {**eeee**} can admit *any* of the 28 possible causal configurations of adjoint 2-planes. (iv) The frames {**ieee**} with dual {**eeee**}, or conversely, may belong also to 28 causal classes, but they do not correspond to the 28 causal configurations of adjoint 2-planes; the number 28 is attained by *different permutations* of some configurations: for example, for the first of these frames the first eight configurations as well as the 11th, 12th, 16th, 17th, 22nd, and 23rd are absent (this arrangement corresponds to that induced by the causal signature, and coincides with the arrangement shown in the first case); in fact, among the 28 causal classes, one half are permutations of the configurations of the other half.

r \ r^*	eeee	ieee	eiee	teee	iiee	tiee	ttee	iiie	tiie	ttie	ttte	iiii	tiii	ttii	ttti	tttt
eeee	++++++,0+++++,-+++++,00++++ -0++++,--++++,000+++,-00+++ --0+++,---+++,0000++,-000++ --00++,---0++,----++,00000+ -0000+,--000+,---00+,----0+ -----+,000000,-00000,--0000 ---000,----00,-----0,------	++++++,0+++++,+++0++,-+++++ 00++++,0++0++,++00++,-0++++ -++0++,++-0++,--++++,000+++ 00+0++,0+00++,-00+++,-0+0++ -+00++,0+-0++,--0+++,--+0++ -+-0++,---+++,0000++,-000++ 00-0++,--00++,-0-0++,---0++		++++++,0+++++ -+++++,00++++ -0++++,--++++ 000+++,-00+++ --0+++,---+++	++++++,0+++++ +0++++,-+++++ 00++++,++00++ +0+0++,-0++++ 00+0++,0+00++ -0+0++,-+00++	++++++ 0+++++ +0++++ -+++++ 00++++ -0++++	++++++ 0+++++ -+++++	++++++ 0+++++ 00++++ 00+0++	++++++ 0+++++ 00++++	++++++ 0+++++	++++++	++++++	++++++	++++++	++++++	++++++
ieee	--0+++,---+++,--00++,--0+0+ ---0++,--0-++,--0+-+,----++ --000+,--0+00,---00+,--0-0+ --00-+,--0+-0,----0+,--0--+ --0+--,-----+,--0000,---000 --0-00,--00-0,----00,--0--0 --00--,-----0,--0---,------	--0+++ ---+++ --00++ --0+0+ ---0++	-00+++ -000++ -00-++ -0000+ -00-0+ -00--+	--0+++ ---+++	-00+++ -000++	-00+++										
teee	---+++,---0++,----++,---00+ ----0+,-----+,---000,----00 -----0,------	---+++, ---0++		---+++												
iiee	--0-0+,--00-+,----0+,-----+ --0-00,--00-0,----00,--00-- --0-0-,-----0,----0-,------	--000+, ---00+			-0000+											
tiee	----0+, -----+, ----00 -----0, ----0-, ------	---00+														
ttee	-----+, -----0, ------															
iiie	--0-00,----00,-----0,------															
tiie	----00, -----0, ------															
ttie	-----0, ------															
ttte	------															
iiii	------															
tiii	------															
ttii	------															
ttti	------															
tttt	------															

Table II. The 199 Causal Classes of Frames of Space-Time.

(c) As shown by Table II, a causal class is, generically, given by the triplet $(\mathbf{r}, \pi, \mathbf{r}^*)$ involving $4+6+4=14$ symbols. The causal signature allows us to condense them: $\sigma(\mathbf{r})$, the ordinal integer of the first column, stands for the four first causal symbols; and this similarly works $\sigma(\mathbf{r}^*)$, except for $\mathbf{r}^*=\{\mathbf{eiee}\}$, for which $\sigma(\mathbf{r}^*)$ is denoted 2_β (see Table II). Due to the features indicated in property (iv) above, a set of indices (say, $a, b, c, \ldots$) is needed to indicate permutations of the same causal configurations of 2-planes. The notation we have adopted is given in Table III. A causal class may thus be indicated by three numbers, a sort of causal coordinates; for example, $(4{:}26_d{:}1)$ stands for $\mathbf{r}=\{\mathbf{iiee}\}$, $\pi=\{--0-0-\}$, $\mathbf{r}^*=\{\mathbf{eeee}\}$.

(d) In paragraph (b) of Section 2 we defined self-dual causal classes. We see now, from Table II, that space-time admits 11 self-dual classes; in causal signature notation they are (1:10:1), (1:13:1), (1:17:1), (1:22:1), (2:10:2), (2:13:2), $(2{:}13_b{:}2)$, $(2{:}13_d{:}2_\beta)$, $(2{:}17{:}2_\beta)$, (3:10:3), and (4:17:4).

(e) As was indicated in the introduction, we are now able to label coordinate systems from the causal point of view. For example, the coordinates (t, x, y, z) for the metric

$$ds^2 = dt^2 + \tfrac{1}{2}\, e^{2y}\, dx^2 - dy^2 - dz^2 + 2\, e^{y}\, dt\, dx$$

(homothetic to the Gödel solution) are not physically admissible; its natural frame $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ belongs to the causal class (6:21:1). Similarly, for coordinates (u, r, θ, ϕ) for the metric

$$ds^2 = A\, du^2 + 2\, du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Table III. Relation Between the Causal Signature and the Causal Character of the Adjoint Planes.[a]

$1 = \{++++++\}$,	$2_a = \{0+++++\}$,	$2_b = \{+++0++\}$,	$2_c = \{+0++++\}$,
$3 = \{-+++++\}$,	$4_a = \{00++++\}$,	$4_b = \{0++0++\}$,	$4_c = \{++00++\}$,
$4_d = \{+0+0++\}$,	$5_a = \{-0++++\}$,	$5_b = \{-++0++\}$,	$5_c = \{++-0++\}$,
$6 = \{--++++\}$,	$7_a = \{000+++\}$,	$7_b = \{00+0++\}$,	$7_c = \{0+00++\}$,
$8_a = \{-00+++\}$,	$8_b = \{-0+0++\}$,	$8_c = \{-+00++\}$,	$8_d = \{0+-0++\}$,
$9_a = \{--0+++\}$,	$9_b = \{--+0++\}$,	$9_c = \{-+-0++\}$,	$10 = \{---+++\}$,
$11 = \{0000++\}$,	$12_a = \{-000++\}$,	$12_b = \{00-0++\}$,	$13_a = \{--00++\}$,
$13_b = \{--0+0+\}$,	$13_c = \{-0-0++\}$,	$13_d = \{-00-++\}$,	$14_a = \{---0++\}$,
$14_b = \{--0-++\}$,	$14_c = \{--0+-+\}$,	$15 = \{----++\}$,	$16 = \{00000+\}$,
$17 = \{-0000+\}$,	$18_a = \{--000+\}$,	$18_b = \{--0+00\}$,	$18_c = \{-00-0+\}$,
$19_a = \{---00+\}$,	$19_b = \{--0-0+\}$,	$19_c = \{--00-+\}$,	$19_d = \{--0+-0\}$,
$19_e = \{-00--+\}$,	$20_a = \{----0+\}$,	$20_b = \{--0--+\}$,	$20_c = \{--0+--\}$,
$21 = \{-----+\}$,	$22 = \{000000\}$,	$23 = \{-00000\}$,	$24 = \{--0000\}$,
$25_a = \{---000\}$,	$25_b = \{--0-00\}$,	$25_c = \{--00-0\}$,	$26_a = \{----00\}$,
$26_b = \{--0--0\}$,	$26_c = \{--00--\}$,	$26_d = \{--0-0-\}$,	$27_a = \{-----0\}$,
$27_b = \{--0---\}$,	$27_c = \{----0-\}$,	$28 = \{------\}$	

[a]The indices differentiate the permutations of the adjoint planes that appear in Table II.

the natural frame $\{\partial_u, \partial_r, \partial_\theta, \partial_\phi\}$ belongs to the causal classes (5:19:2), (4:17:4), or (2:8:5), depending on whether $A>0$, $A=0$, or $A<0$ [the well-known Vaidya solution corresponds to $A=1-2m(u)/r$]. This shows that the unusual character of a coordinate system may be quantitatively characterized.

(f) Table II may be considered as a sort of graphic representation of a *theorem of signature.* The hyperbolic character and the sign of signature of a regular matrix can be obtained from the table by analyzing the sign of the second-order principal minors (they are nothing but the causal signs) and their compatibility with the signs of the terms of the principal diagonals of the matrix and its inverse. For example, if a matrix having $a_{\alpha\alpha}=0$ is to be hyperbolic, the second-order principal minors must all be strictly negative and then we know that all the elements $a^{\alpha\alpha}$ of its inverse will have the same sign ε; it follows that the signature of the metric is $(-\varepsilon, \varepsilon, \varepsilon, \varepsilon)$, as corresponds to the case {**iiii**, ------, **eeee**} of the table.

(g) Let us note that, except for the cases (3:10:3) and (4:17:4), the boundary of the impossible Lorentzian frames is a "convex" stair; these two distinct cases correspond to the physically admissible and null frames. What is the role of this "peninsular" isolation in our inertia to conceive physically other different frames?

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